Time Series Analysis

Basics

$$\gamma_0 = \frac{1}{N}\sum_{t=1}^{N}(x_t - \bar{x})^2$$

$$\gamma_k = \frac{1}{N}\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{t+k} - \bar{x})$$

$$\rho_0 = 1$$

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

$$\hat{\sigma}^2_X = \frac{\hat{\sigma}^2}{n}\left(1 + 2\sum_{k=1}^{n-1}\left(1 - \frac{|k|}{n}\hat{\rho_k}\right)\right)$$

$$CI : \bar{X} \pm 1.96\sqrt{\frac{\hat{\sigma}^2}{n}\left(1 + 2\sum_{k=1}^{n-1}\left(1 - \frac{|k|}{n}\hat{\rho_k}\right)\right)}$$

Autoregressive Models

AR(1) Models

$$X_t - \phi X_{t-1} = a_t$$

$$(1 - \phi z) = 0$$

$$\rho_0 = 1$$

$$\rho_k = \phi_1^k$$

$$\sigma^2_X = \frac{\sigma^2_a}{1 - \phi_1^2}$$

AR(1) Properties

- Positive ϕ
  - Realizations appear to be wandering (aperiodic)
  - Autocorrelations are damped exponentials
  - Spectral densities have peaks at zero
- Negative ϕ
  - Realizations appear to be oscillating
  - Autocorrelations are damped oscillating exponentials
  - Spectral densities have peaks at $f = 0.5$

AR(2) Models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t$$

$$(1 - \phi_1 z - \phi_2 z^2) = 0$$

$$\rho_0 = 1$$

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

$$\rho_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1 - \phi_2}$$

$$\sigma^2_X = \frac{1}{1 - \phi_1\rho_1 - \phi_2\rho_2}$$

AR(2) Properties

- Two Real Roots - Both Pos
  - The realization will appear to be wandering
  - The autocorrelations will be exponentially damped
  - There will be a peak at 0
- Two Real Roots - Both Neg
  - The realization will appear to be oscillating
  - The autocorrelations will be damped oscillating exponentials
  - There will be a peak at 0.5
- Two Real Roots - One Each
  - The realization will appear to be wandering and an oscillation will run on the realization
  - The autocorrelations will be exponentially damped with a hint of oscillation
  - There will be peaks at 0 and 0.5 in the spectal density
- One Complex
  - The realization will appear to have a pseudo-cyclic behavior with a cycle length of $\frac{1}{f_0}$
  - The autocorrelations will be damped exponentials oscillating in a sinusoid envelope with a frequency of f_0
  - There will be a peak at f_0 (between 0 and 0.5)

$$f_0 = \frac{1}{2\pi}cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right)$$

AR(p) Models

$$X_t - \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_2 X_{t-p} = a_t$$

$$x_t - \phi_1 B X_t - \phi_2 B^2 X_t - ... - \phi_p B^p X_t = a_t$$

Key Concepts

- An AR(p) model is stationary if and only if all the roots of the characteristic equation are outside the unit circle.
- Any AR(p) characteristic equation can be numerically factored into 1st and 2nd order elements.
- These factors are interpreted as contributing AR(1) and AR(2) behaviors to the total behavior of the AR(p) model.

Factor Contributions

AR(p) models reflect a contribution of AR(1) and AR(2) contributions. Roots that are close to the unit circle will be the dominate behavior.

- First order factors $(1 - \phi_1 B)$
  - Associated with real roots
  - Contribute AR(1)-type behavior to the AR(p) model
  - Associated with a system frequency of 0 if ϕ_1 is positive or 0.5 if ϕ_1 is negative
- Second order factors $(1 - \phi_1 B - \phi_2 B^2)$
  - Associated with complex roots
  - Contribute cyclic AR(2) behavior to the AR(p) model
  - Associated with a system frequency of f_0

Moving Average Models

MA(1) Models

$$X_t = a_t - \theta a_{t-1}$$

$$(1 - \theta_1 z) = 0$$

$$\rho_0 = 1$$

$$\rho_1 = \frac{-\theta_1}{1 + \theta_1^2}$$

$$\rho_k = 0|_{k>1}$$

$$\sigma^2_X = \sigma^2_a\left(1 + \theta_1^2\right)$$

MA(2) Models

$$X_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$

$$(1 - \theta_1 z - \theta_2 z^2) = 0$$

$$\rho_0 = 1$$

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_k = 0|_{k>2}$$

$$\sigma^2_X = \sigma^2_a\left(1 + \theta_1^2 + \theta_2^2\right)$$

MA(q) Models

$$X_t = a_t - \theta_1 a_{t-1} - ... - \theta_2 a_{t-q}$$

$$x_t = a_t - \theta_1 B a_t - ... - \theta_q B^q X_t$$

Key Concepts

- MA models are a finite GLP
- MA models are always stationary
- MA models are invertable iff all the roots are outside of the unit circle.

MA Inversion

- Real Root: use $1/\theta$
- Complex Roots: use $\theta_1 = r_1^{-1} + r_2^{-1}$ and $\theta_2 = -r_1^{-1} r_2^{-1}$

ARMA(p,q) Models

$$X_t = \beta + \phi_1 X_{t-1} + ... + \phi_2 X_{t-p} = a_t - \theta_1 a_{t-1} - ... - \theta_2 a_{t-q}$$
$$x_t - \phi_1 B X_t - ... - \phi_p B^p X_t = a_t - \theta_1 B a_t - ... - \theta_q B^q X_t$$

Key Concepts

- Valid when the model is stationary and invertable
  - Stationary: roots of $\phi(z)$ are outside the unit circle
  - Invertable: roots of $\theta(z)$ are outside the unit circle
- $\phi(z)$ and $\theta(z)$ have no common factors (check)

ARIMA

General Form

$$\phi(B)(1-B)^d X_t = \theta(B) a_t$$

Properties

- The roots on the unit circle dominate the behavior of the realization
- The autocorrelations are defined to have a magnitude of 1 ($\rho_k = 1$)
- The variance of ARIMA is not well defined

ARUMA

ARUMA is an generalization of ARIMA that includes a term or term(s) for seasonality.

$$\phi(B)(1-B)^d (1-B^s) X_t = \theta(B) a_t$$

Monthly Seasonality

$$\left(1-B^4\right) = (1-B)(1+B)\left(1+B^2\right)$$

General Linear Processes

General Form

Use `psi.weights.wge` to calculate ψs

$$X_t - \mu = \sum_{j=0}^{\infty} \psi_j a_{t-j}$$

- An MA model can be represented as finite GLP
- An AR model can be represented as infinite GLP

Forecasting

Notation

- t_0 - origin of the forecast
- l - number of time units to forecast (lead time)
- $\hat{X}_{t_0}(l)$ - the forecast of X_{t_0+l} given data up to t_0

ARMA Forecasting

Use `fore.arma.wge()` for forecasting.

$$\hat{X}_{t_0}(l) = \sum_{i=1}^{p} \phi_i \hat{X}_{t_0}(l-i) - \sum_{j=1}^{q} \theta_j \hat{a}_{t_0+l-j} + \bar{x}\left[1 - \sum_{i=1}^{p} \phi_i\right]$$

$$\hat{\sigma}_a^2 = \frac{1}{n-p} \sum_{t=p+1}^{n} \hat{a}_t^2$$

Facts

$$e_{t_0}(l) = X_{t_0+l} - \hat{X}_{t_0}(l)$$
$$var\left[e_{t_0}(l)\right] = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2$$
$$FI : \hat{X}_{t_0}(l) \pm z_{1-\alpha/2} \sigma_a \left[\sum_{k=0}^{l-1} \psi_k^2\right]^{1/2}$$

ARIMA Forecasting

Use `fore.aruma.wge()` for forecasting.

- Limits become unbounded as l increases
- A factor of $(1-B)$ does not forecast a trend. An order of $d > 1$ is required to forecast a trend.

ARIMA with Seasonality Forecasting

The forecast for step l is same as the last s value. Use `fore.aruma.wge()` for forecasting.

- Limits become unbounded as l increases
- A factor of $(1-B)$ does not forecast a trend. An order of $d > 1$ is required to forecast a trend.
- $(1-B)(1-B^s) = a_t$ is called an airline model.

Linear Forecasting

Use `fore.sigplusnoise.wge()` for forecasting.

- Fit an OLS to X_t
- Fit an AR(p) to the residuals (Z_t)

$$\hat{X}_{t_0}(l) = b_0 + b_1 t + \hat{Z}_{t_0}(l)$$

$$FI : b_0 + b_1 t + \hat{Z}_{t_0}(l) \pm z_{1-\alpha/2} \hat{\sigma}_a \left[\sum_{k=0}^{l-1} \psi_k^2\right]^{1/2}$$

Non-Stationary Tests

Dicky-Fuller Test

$$H_0 : \textit{The model has a root of } +1$$
$$H_a : \textit{The model does not have a root of } +1$$

This test has a high type II error rate, increasing as a root approaches the unit circle.

Cochrane-Orcutt Test

This is a test for the presence of a linear slope corrected for an AR(1) noise structure.

$$H_0 : b = 0$$
$$H_a : b \neq 0$$

Metrics

AIC - ARMA Objective

$$AIC = ln(\hat{\sigma}_a) + 2\left(\frac{p+q+1}{n}\right)$$

Filtering

Filters transform time series.

$$Z_t \rightarrow H(B) \rightarrow X_t$$

$$X(t) = Z(t)H(B)$$

There are four basic types of filters.

- High pass - filters out low frequencies
- Low pass - filters out high frequencies
- Band pass - filters out frequencies outside the band
- Band stop - filters out frequencies inside the band

Difference Filter

The first order difference is expressed by the following

$$X_t = Z_t - Z_{t-1}$$
$$H(B) = B^0 - B$$

This is a high pass filter.

Moving Average Filter

A 5-point moving average filter can be expressed as

$$X_t = \frac{Z_{t+2} + Z_{t+1} + Z_t + Z_{t-1} + Z_{t-2}}{5}$$
$$H(B) = \frac{B^{-2} + B^{-1} + B^0 + B + B^2}{5}$$

This is a low pass filter.

Band-Type Filter

High pass and low pass filters can be combined to produce band pass and band stop filters.